

LIAPOUNOFF'S THEOREM FOR NONATOMIC, FINITELY-ADDITIVE, BOUNDED, FINITE-DIMENSIONAL, VECTOR-VALUED MEASURES

BY

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ABSTRACT. Liapounoff's theorem states that if (X, Σ) is a measurable space and $\mu: \Sigma \rightarrow \mathbf{R}^d$ is nonatomic, bounded, and countably additive, then $\mathcal{R}(\mu) = \{\mu(A): A \in \Sigma\}$ is compact and convex. When Σ is replaced by a σ -complete Boolean algebra or an F -algebra (to be defined) and μ is allowed to be only finitely additive, $\mathcal{R}(\mu)$ is still convex. If Σ is any Boolean algebra supporting nontrivial, nonatomic, finitely-additive measures and Z is a zonoid, there exists a nonatomic measure on Σ with range dense in Z . A wide variety of pathology is examined which indicates that ranges of finitely-additive, nonatomic, finite-dimensional, vector-valued measures are fairly arbitrary.

1. Ranges need not be compact. Liapounoff's theorem, as proved by Halmos [10] or Lindenstrauss [13], state that if Σ is a σ -algebra of subsets of a set X and $\mu: \Sigma \rightarrow \mathbf{R}^d$ is a nonatomic countably additive function then the range, $\mathcal{R}(\mu) = \{\mu(A): A \in \Sigma\}$, of μ is a convex compact set. We are interested in the finitely additive analogue. Here Σ is replaced by some Boolean algebra \mathcal{B} with supremum X and $\mu: \mathcal{B} \rightarrow \mathbf{R}^d$ denoted by (μ_1, \dots, μ_d) where μ_j , the j th component of μ , is in $BA(\mathcal{B})$, the finitely-additive measures on \mathcal{B} of bounded variation, and is nonatomic for all j . Recall [1], [14], [20], [21] that a *nonatomic* measure μ on \mathcal{B} is one such that for any $\varepsilon > 0$ there is a partition $\{A_1, \dots, A_n\}$ of \mathcal{B} such that the total variation, $|\mu|(A_j)$, of A_j under μ is at most ε . Such measures are also called continuous. A measure μ on \mathcal{B} is nonatomic iff the corresponding Radon measure $\tilde{\mu}$ on the Stone space $X_{\mathcal{B}}$ is a nonatomic Radon measure. All measures absolutely continuous with respect to nonatomic measures are again nonatomic, both in $\mathcal{M}(X_{\mathcal{B}})$, the Radon measures on $X_{\mathcal{B}}$, and in $BA(\mathcal{B})$. If μ is a nonatomic vector-valued measure on \mathcal{B} with values in \mathbf{R}^d and l is any linear functional on \mathbf{R}^d , then $l \circ \mu$, defined by composition, is a nonatomic element of $BA(\mathcal{B})$. More generally, l may be a linear transformation of \mathbf{R}^d into another vector space E , in which case $l \circ \mu$ is a nonatomic vector-valued measure with finite-dimensional range in E .

Liapounoff's theorem fails in the finitely-additive context only in the fact that $\mathcal{R}(\mu)$ need not be compact. Due to lack of compactness Lindenstrauss' elegant proof cannot be adapted to show convexity of $\mathcal{R}(\mu)$. An adaptation of Halmos' proof is available to establish this result (Theorem 2-2).

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LEMMA 1-1. Let $\mu: \mathfrak{B} \rightarrow \mathbb{R}^d$ be a bounded nonatomic measure on \mathfrak{B} . $\overline{\mathcal{R}(\mu)}$ is convex and compact. If $\mu = (\mu_1, \dots, \mu_d)$ and $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_d)$ is the corresponding Baire measure on the Stone space $X_{\mathfrak{B}}$, then $\overline{\mathcal{R}(\mu)} = \mathcal{R}(\tilde{\mu}) = \{\tilde{\mu}(A): A \text{ Baire } \subset X_{\mathfrak{B}}\}$.

PROOF. There is a bijection $A \rightarrow [A]$ from \mathfrak{B} to clopen sets of $X_{\mathfrak{B}}$. For any $A \in \mathfrak{B}$, $\mu(A) = \tilde{\mu}([A])$. Since clopen sets in $X_{\mathfrak{B}}$ are Baire, $\mathcal{R}(\mu) \subset \mathcal{R}(\tilde{\mu})$. Liapounoff's theorem guarantees that $\mathcal{R}(\tilde{\mu})$ is convex and compact, hence contains $\overline{\mathcal{R}(\mu)}$. Since $X_{\mathfrak{B}}$ is totally disconnected the Baire algebra is the monotone sequential closure of the clopen algebra. Let \mathcal{E} denote those Baire sets $E \subset X_{\mathfrak{B}}$ with $\tilde{\mu}(E) \in \mathcal{R}(\mu)$. \mathcal{E} contains the clopen algebra of $X_{\mathfrak{B}}$. If $\{E_n: n \in N\}$ is a monotone sequence in \mathcal{E} with limit E , then $\tilde{\mu}(E) = \lim_{n \rightarrow \infty} \tilde{\mu}(E_n) \in \overline{\mathcal{R}(\mu)}$. Thus, \mathcal{E} is monotone sequentially closed, hence is the Baire algebra. This establishes the lemma. \square

REMARKS. (1) This is the strongest version of Liapounoff's theorem valid for general Boolean algebras. This is easily seen by considering the case where $\text{card}(\mathfrak{B}) = \aleph_0$, in which case $\mathcal{R}(\mu)$ is also countable.

(2) For a Boolean algebra \mathfrak{B} to carry a nontrivial nonatomic measure it is necessary and sufficient that \mathfrak{B} contain a countable nonatomic subalgebra \mathfrak{B}_0 (i.e., the clopen algebra of the Cantor set) [16, Theorem 1]. It is only in this context that the considerations in this article are nonvacuous.

COROLLARY 1-1-1. Let $\mu = (\mu_1, \mu_2, \dots, \mu_d)$ be a nonatomic measure on the Boolean algebra \mathfrak{B} . In order that $\{\mu_1, \dots, \mu_d\}$ be a mutually singular set of measures on \mathfrak{B} it is necessary and sufficient that $\overline{\mathcal{R}(\mu)} = \prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$.

PROOF. We always have $-\|\mu_j^-\| \leq \mu_j(A) \leq \|\mu_j^+\|$ when $A \in \mathfrak{B}$. Hence, $\mathcal{R}(\mu)$ and, consequently, $\overline{\mathcal{R}(\mu)}$ are in $\prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$. The measures $\{\mu_1, \dots, \mu_d\}$ are mutually singular iff there exists a partition $\{A_1^+, A_1^-, \dots, A_d^+, A_d^-\}$ such that $\|\mu_j^+\| - \varepsilon \leq \mu_j(A_j^+) \leq \|\mu_j^+\|$ and $\|\mu_j^-\| - \varepsilon \leq -\mu_j(A_j^-) \leq \|\mu_j^-\|$ for all $j = 1, \dots, d$. By letting $A = \bigcup_{j=1}^d A_j^{\varepsilon(j)}$, where $\varepsilon: \{1, \dots, d\} \rightarrow \{+, -\}$ is arbitrarily chosen, we find that $\mu(A)$ is arbitrarily close to one of the corners of the rectangular parallelepiped $\prod_{j=1}^d [-\|\mu_j^-\|, \|\mu_j^+\|]$, hence each corner is in $\overline{\mathcal{R}(\mu)}$. Since $\overline{\mathcal{R}(\mu)}$ is convex the entire parallelepiped is in $\overline{\mathcal{R}(\mu)}$ if $\{\mu_1, \dots, \mu_d\}$ are mutually singular. Now suppose that μ is such that $\overline{\mathcal{R}(\mu)}$ is the entire parallelepiped. There is for any $i \neq j$ an $A \in \mathfrak{B}$ such that $\mu_i(A) \geq \|\mu_i^+\| - \varepsilon$ and $\mu_j(A) \geq \|\mu_j^+\| - \varepsilon$. We have then that $\mu_i^+(A) \geq \|\mu_i^+\| - \varepsilon$ and $\mu_j^-(A) \leq \varepsilon$. Since ε is arbitrary, $\mu_i^+ \perp \mu_j^-$. Similarly $\mu_i^+ \perp \mu_j^+$, $\mu_i^- \perp \mu_j^-$, and $\mu_i^- \perp \mu_j^+$. Thus $\mu_i \perp \mu_j$. Since $i \neq j$ are arbitrary the corollary is established. \square

THEOREM 1-2. If \mathfrak{B} is a Boolean algebra admitting a nontrivial nonatomic measure there exists a $\mu: \mathfrak{B} \rightarrow \mathbb{R}^2$ which is nonatomic, nonnegative and finitely additive with $\mathcal{R}(\mu) \neq \overline{\mathcal{R}(\mu)}$.

PROOF. Assume to the contrary that $\mathcal{R}(\mu)$ is compact for all finitely-additive, nonatomic, nonnegative $\mu: \mathfrak{B} \rightarrow \mathbb{R}^2$. Let μ be a nonatomic element of $BA(\mathfrak{B})$ and let $\mu' = (\mu^+, \mu^-)$. There exists, by Corollary 1-1-1, an $A \in \mathfrak{B}$ such that $\mu^+(A) = \|\mu^+\|$ and $\mu^-(A) = 0$. As a consequence, $\mu(A) = \sup\{\mu(A'): A' \in \mathfrak{B}\} = \|\mu^+\|$.

Sobczyk and Hammer [20, Corollary 2.1] implies that μ restricted to A is μ^+ and μ restricted to A^c is $-\mu$. That is, $\mu(A') = \mu^+(A' \cap A) - \mu^-(A' \setminus A)$ for all A' . Thus, μ has a Hahn decomposition for all nonatomic $\mu \in BA^+(\mathfrak{B})$. As a consequence, if μ_1 and μ_2 are singular elements of $BA^+(\mathfrak{B})$ there is an $A \in \mathfrak{B}$ such that $\mu_1(A) = \|\mu_1\|$ and $\mu_2(A) = 0$, for if $\mu = \mu_1 - \mu_2$, then $\mu^+ = \mu_1$ and $\mu^- = \mu_2$. If $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are mutually singular elements of $\mathfrak{N}^+(X_{\mathfrak{B}})$, then there exists a clopen subset A of $X_{\mathfrak{B}}$ such that $\text{supp}(\tilde{\mu}_1) \subset A$ and $\text{supp}(\tilde{\mu}_2) \subset A^c$. Pick a nontrivial $\mu \in BA^+(\mathfrak{B})$ which is nonatomic. Let $\tilde{\mu}$ be the corresponding measure on $X_{\mathfrak{B}}$ and let $X_{\mu} = \text{supp}(\tilde{\mu})$. Since $\tilde{\mu} \neq 0$ is a nonatomic element of $\mathfrak{N}^+(X_{\mu})$; X_{μ} is not a scattered compact Hausdorff space (Semadeni [19, 19.7.6]). Since X_{μ} is nonscattered there is a closed separable perfect subset $Y \neq \emptyset$ such that $\tilde{\mu}(Y) = 0$ (Semadeni [19, 19.7.8(c)]). Since Y is nonscattered there exists a $\nu \in \mathfrak{N}^+(Y)$ which is nonatomic and nontrivial. Consider $\tilde{\nu}$ as an element of $\mathfrak{N}^+(X_{\mathfrak{B}})$ and let ν be the corresponding nonzero nonatomic element of $BA^+(\mathfrak{B})$. Since $\tilde{\mu}(Y) = 0$ and $\tilde{\nu}(Y) = \|\nu\|$, $\tilde{\mu}$ and $\tilde{\nu}$ are singular as are μ and ν . By assumption there is an $A \in \mathfrak{B}$ with $\mu(A) = 0$ and $\nu(A) = \|\nu\|$. If $[A]$ is the corresponding clopen subset of $X_{\mathfrak{B}}$ we have $Y \subset [A]$ and $X_{\mu} \cap [A] \neq \emptyset$, which is impossible, since $Y \subset X_{\mu}$. This contradicts our assumption that $\mathfrak{R}(\mu)$ is compact for all $\mu: \mathfrak{B} \rightarrow \mathbf{R}^2$ which are nonnegative and nonatomic. \square

COROLLARY 1-2-1. *If \mathfrak{B} supports a nontrivial, nonatomic measure there is a nonatomic, nonnegative $\mu: \mathfrak{B} \rightarrow \mathbf{R}^2$ with singular components $\mu_1 \neq 0$ and $\mu_2 \neq 0$ so that $(\|\mu_1\|, 0)$ and $(0, \|\mu_2\|)$ are in $\overline{\mathfrak{R}(\mu)} \setminus \mathfrak{R}(\mu)$.*

COROLLARY 1-2-2. *With the same assumptions on \mathfrak{B} there exists $\mu \in BA(\mathfrak{B})$ which is nonatomic with nontrivial positive and negative parts such that $\mathfrak{R}(\mu) = (-\|\mu^-\|, \|\mu^+\|)$.*

COROLLARY 1-2-3. *With the same assumptions on \mathfrak{B} there is an uncountable family $\{\mu_{\alpha}\} \subset BA_1^+(\mathfrak{B})$ of nonatomic probabilities such that $\mu_{\alpha} \perp \mu_{\beta}$ if $\alpha \neq \beta$.*

PROOF. The proof of Theorem 1-2 demonstrated the existence, for any μ nonatomic on \mathfrak{B} , of a nonatomic ν with $\nu \perp \mu$. Let $\{\mu_{\alpha}\}$ be a maximal mutually singular collection in $BA_1^+(\mathfrak{B})$ consisting of nonatomic measures. If this collection is countable let μ be a countable convex combination so that $\mu_{\alpha} \ll \mu$ for all α . There is a ν nontrivial and nonatomic with $\nu \perp \mu$, hence with $\nu \perp \mu_{\alpha}$ for all α . Since ν may be chosen in $BA_1^+(\mathfrak{B})$, the maximality of $\{\mu_{\alpha}\}$ is contradicted. Thus, $\{\mu_{\alpha}\}$ is uncountable. \square

Recall that a nonnegative measure space (X, Σ, μ) is *semifinite* iff for any $A_0 \in \Sigma$ with $\mu(A_0) > 0$, one has $\mu(A_0) = \sup\{\mu(A): A \subset A_0, \mu(A) < \infty\}$. Also recall that $L^{\infty*+}(X, \Sigma, \mu)$ is the positive cone of the dual of $L^{\infty}(X, \Sigma, \mu)$.

COROLLARY 1-2-4. *Let (X, Σ, μ) be a nonnegative semifinite measure space. There exists in $L^{\infty*+}(X, \Sigma, \mu)$ an uncountable family of elements singular with respect to $L^1(X, \Sigma, \mu)$ provided that $L^1(X, \Sigma, \mu)$ is infinite dimensional.*

PROOF. By restriction to an element A of Σ with $0 < \mu(A) < \infty$, one may assume that μ is a finite nonatomic measure on (X, Σ) . $L^{\infty*+}(X, \Sigma, \mu)$ may be considered

as $BA^+(\Sigma_\mu)$ where Σ_μ is the quotient of Σ modulo μ -negligible sets. Apply Corollary 1-2-3 to $\mathfrak{B} = \Sigma_\mu$. If this assumption is not valid then there is a σ -finite $A \in \Sigma$ which is a union of \aleph_0 μ -atoms. By restriction of μ to A one can assume that $L^\infty(X, \Sigma, \mu)$ is Banach lattice isomorphic to $l^\infty \cong C(\beta N)$. Since βN supports nonatomic Radon probabilities, Corollary 1-2-3 is once again applicable since there is a nonatomic probability measure on Σ_μ . \square

REMARKS. (1) In Corollaries 1-2-3 and 1-2-4 one can obtain mutually singular families of cardinality c and 2^c , respectively, where $c = 2^{\aleph_0}$. See Corollaries 3-2-2 and 3-2-3.

(2) Do there exist uncountable mutually singular families of countably-additive nonatomic probabilities on a measurable space (X, Σ) ? When Σ is only required to be a σ -complete Boolean algebra, the answer is no. Take (X, Σ, μ) to be a nonatomic probability space. All countably additive measures on Σ_μ , the quotient of Σ with the μ -negligible sets, are absolutely continuous with respect to μ and the countable chain conditions on Σ_μ rules out having uncountably many, disjoint, countably-additive probabilities on Σ_μ . If Σ is the Baire algebra of a nonscattered compact Hausdorff space there do exist such families. One instance of a measurable space (X, Σ) not admitting such a family is when Σ contains all sets of outer measure 0 for some probability measure μ . In this case only countably many members of a mutually singular family of countably-additive probabilities on Σ are not singular with respect to μ . If the family is uncountable there is a μ -negligible set A and a nonatomic countably-additive probability ν with $\nu(A) = 1$. In this case ν is a nonatomic countably-additive probability defined on 2^A , hence $\text{card}(A)$ is a real-valued measurable cardinal [3]. The converse is also true.

2. Ranges are convex. Let \mathfrak{B} be an F -algebra, which is defined to be an algebra whose Stone space is an F -space in the sense of Gillman and Jerison [8], [18]. Seever shows that \mathfrak{B} is an F -algebra iff whenever $\{A_n\}$ is an increasing sequence and $\{B_n\}$ is a decreasing sequence in \mathfrak{B} with $A_n \subset B_n$ for all n , there is a $C \in \mathfrak{B}$ with $A_n \subset C \subset B_n$ for all n . This may serve as our definition of an F -algebra. Any σ -complete algebra \mathfrak{B} is an F -algebra. Seever shows that any closed subspace of an F -space is an F -space, hence it follows that any quotient algebra \mathfrak{B}/η via an ideal η is an F -algebra if \mathfrak{B} is an F -algebra. The chief means of constructing F -algebras is as quotients of σ -complete algebras. Under the assumption of Martin's Axiom and $c = \aleph_2$, van Douwen and van Mill [22] have constructed an F -algebra not arising in this fashion.

We shall need the following lemma, which is the one-dimensional version of Liapounoff's theorem on F -algebras.

LEMMA 2-1 (MAHARAM [14]). *Let \mathfrak{B} be an F -algebra and let $\mu \in BA^+(\mathfrak{B})$. $\mathcal{R}(\mu) = [0, \|\mu\|]$ if μ is nonatomic.*

PROOF. Suppose we have found sequences $A_1 \subset A_2 \subset \cdots \subset A_n \subset B_n \subset \cdots \subset B_1$ in \mathfrak{B} so that $\lambda - 2^{-n} \leq \mu(A_n) \leq \mu(B_n) \leq \lambda + 2^{-n}$ where $\lambda \in (0, \|\mu\|)$ is fixed beforehand. Find a partition $\{C_1, \dots, C_n\}$ of $B_n \setminus A_n$ into sets of

μ measure at most 2^{-n-1} . Let m_0 be the last integer, possibly 0, so that $\mu(A_n) + \sum_{i=1}^{m_0} \mu(C_i)$ is less than or equal to λ . Set $A_{n+1} = A_n \cup \bigcup_{i=1}^{m_0} C_i$ and set $B_{n+1} = A_n \cup \bigcup_{i=1}^{m_0+1} C_i$. We have $A_n \subset A_{n+1} \subset B_{n+1} \subset B_n$ and $\lambda - 2^{-n-1} < \mu(A_{n+1}) < \mu(B_{n+1}) \leq \lambda + 2^{-n-1}$. Find a $C \in \mathfrak{B}$ satisfying $A_n \subset C \subset B_n$ for all n . We have $\mu(C) = \lambda$. Since $\lambda \in (0, \|\mu\|)$ is arbitrary, $\mathfrak{R}(\mu) = [0, \|\mu\|]$. \square

REMARK. (1) Obvious modifications of the proof of Lemma 2-1 demonstrates that if μ is a finite-dimensional, nonatomic, nonnegative, finitely-additive measure on the F -algebra \mathfrak{B} , then, when $\{x^1, x^2\} \subset \mathfrak{R}(\mu)$, there exists a continuous path in $\mathfrak{R}(\mu)$ starting at x^1 and ending at x^2 . This hints that $\mathfrak{R}(\mu)$ is convex.

(2) Lemma 2-1 has been established by Granirer [9] for left invariant means on certain infinite semigroups. When suitably interpreted, L. J. Savage [17, Theorem 3.3.2], establishes Lemma 2-1 for nonatomic measures on the power set of a discrete set.

THEOREM 2-2. Let μ be a finite-dimensional, nonatomic, bounded, finitely-additive measure on the F -algebra \mathfrak{B} . $\mathfrak{R}(\mu)$ is convex.

PROOF. We first assume that the theorem is true for all n -dimensional nonnegative measures μ and demonstrate its validity for $(n+1)$ -dimensional measures. By induction and Lemma 2-1 the theorem is true for all finite dimensional nonnegative μ . To obtain the theorem for a general $(\mu_1, \dots, \mu_n) = \mu$, one appeals to its validity for $(\mu_1^+, \mu_1^-, \dots, \mu_n^+, \mu_n^-) = \nu$ and the fact that $\mathfrak{R}(\mu)$ and $\mathfrak{R}(\bar{\mu})$ are the linear images of $\mathfrak{R}(\nu)$ and $\mathfrak{R}(\bar{\nu})$ under the obvious transformations.

Let $(\mu_1, \dots, \mu_n, \mu_{n+1}) = \mu$ be a nonnegative, nonatomic measure on \mathfrak{B} such that μ_{n+1} is absolutely continuous with respect to $(\mu_1, \dots, \mu_n) = \mu'$, so that if $\{A_n\}$ is a sequence in \mathfrak{B} with $\mu'(A_n) \rightarrow 0$ then $\mu_{n+1}(A_n) \rightarrow 0$. By inductive assumption $\mathfrak{R}(\mu')$ is convex. For any $E \in \mathfrak{B}$ there is an $E_{1/2} \subset E = E_1$ with $\mu'(E_{1/2}) = \frac{1}{2}\mu'(E)$. There are, as a result, $E_{1/4}$ and $E_{3/4}$ with $E_{1/4} \subset E_{1/2}$, with $E_{1/2} \subset E_{3/4} \subset E_1$, with $\mu'(E_{1/4}) = \frac{1}{4}\mu'(E)$ and $\mu(E_{3/4}) = \frac{3}{4}\mu'(E)$. By induction, for all dyadic rationals k in $[0, 1]$, there is an E_k satisfying $E_k \subset E_{k'}$ for $k < k'$ with k' dyadic and satisfying $\mu'(E_k) = k\mu'(E)$. Here $E_0 = \emptyset$. Since \mathfrak{B} is an F -algebra there exist $\{E_\lambda: \lambda \in [0, 1]\}$, with $E_\lambda \subset E_{\lambda'}$ if $\lambda \leq \lambda'$, and $\mu'(E_\lambda) = \lambda\mu'(E)$. If E and F are disjoint let $\{E_\lambda: \lambda \in [0, 1]\}$ and $\{F_\lambda: \lambda \in [0, 1]\}$ be the corresponding families obtained by the previous construction. If $C_\lambda = E_{1-\lambda} \cup F_\lambda$ then $\mu'(C_\lambda) = (1-\lambda)\mu'(E) + \lambda\mu'(F)$ for all $\lambda \in [0, 1]$, $C_0 = E$, and $C_1 = F$. We have

$$C_\lambda \triangle C_\delta = (F_\lambda \setminus F_\delta) \cup (E_{1-\delta} \setminus E_{1-\lambda}) \quad \text{if } \lambda \geq \delta$$

so $\mu'(C_\lambda \triangle C_\delta) \rightarrow 0$ if $\lambda - \delta \rightarrow 0$. If E and F are not disjoint, replace them by $E \setminus F$ and $F \setminus E$ in the above construction. To each C_λ obtained adjoin the disjoint set $E \cap F$ to get a new C_λ . Again $\mu'(C_\lambda) = (1-\lambda)\mu'(E) + \lambda\mu'(F)$ and if $\lambda - \delta \rightarrow 0$ then $\mu'(C_\lambda \triangle C_\delta) \rightarrow 0$. Consequently, $\mu_{n+1}(C_\lambda \triangle C_\delta) \rightarrow 0$ if $\lambda - \delta \rightarrow 0$.

Let $A \in \mathfrak{B}$. We will show that there is an $A_{1/2} \subset A$ with $\mu(A_{1/2}) = \frac{1}{2}\mu(A)$. If this is the case for all $A \in \mathfrak{B}$, imitation of the previous paragraph shows that if $\{E, F\} \subset \mathfrak{B}$, there is a family $\{C_\lambda: \lambda \in [0, 1]\}$ such that $C_0 = E$, $C_1 = F$, $\mu(C_\lambda) = (1-\lambda)\mu(E) + \lambda\mu(F)$ for $0 \leq \lambda \leq 1$, and $\mu(C_\lambda \triangle C_\delta) \rightarrow 0$ as $\lambda - \delta \rightarrow 0$. Thus, in this case, $\mathfrak{R}(\mu)$ is convex. There is an $E \subset A$ with $\mu'(E) = \frac{1}{2}\mu'(A)$. Let $F = A \setminus E$

so $\mu'(F) = \frac{1}{2}\mu'(A)$. Let $\{C_\lambda\}$ be the family constructed in the previous paragraph with $\mu'(C_\lambda) = (1 - \lambda)\mu'(E) + \lambda\mu'(F) = \frac{1}{2}\mu'(A)$. Since μ_{n+1} is absolutely continuous with respect to μ' , the function $\lambda \rightarrow \mu_{n+1}(C_\lambda)$ is continuous in λ . If $\mu_{n+1}(E) > \frac{1}{2}\mu_{n+1}(A)$, then $\mu_{n+1}(C_0) > \frac{1}{2}\mu_{n+1}(A) > \mu_{n+1}(C_1)$, otherwise $\mu_{n+1}(C_0) < \frac{1}{2}\mu_{n+1}(A) < \mu_{n+1}(C_1)$. In any case there is a λ such that $\mu_{n+1}(C_\lambda) = \frac{1}{2}\mu_{n+1}(A)$. Let $A_{1/2} = C$ for this λ . Thus, when μ_{n+1} is absolutely continuous with respect to μ' , $\mathcal{R}(\mu)$ is convex.

Now suppose that $(\mu_1, \dots, \mu_{n+1})$ is any nonatomic, nonnegative, finitely-additive measure on \mathcal{B} . Set $\nu = \sum_{i=1}^{n+1} \mu_i$. The measure $\nu = (\nu_1, \dots, \nu_{n+1})$ is nonatomic and has ν_{n+1} absolutely continuous with respect to (ν_1, \dots, ν_n) . As in Halmos [10, Lemma 6], $\mathcal{R}(\mu)$ is a 1-1 linear image of $\mathcal{R}(\nu)$. Since $\mathcal{R}(\nu)$ is convex, $\mathcal{R}(\mu)$ is also convex. This establishes that the range of any finite-dimensional vector measure is convex. \square

COROLLARY 2-2-1. (a) Let \mathcal{B} be an F -algebra and μ a nonnegative and nonatomic finitely-additive map from \mathcal{B} to R^d with $d < \infty$. For any $E \in \mathcal{B}$ there is an element f_E of the continuous functions $\mathcal{C}(X_{\mathcal{B}})$ on the Stone space $X_{\mathcal{B}}$ of \mathcal{B} such that $0 \leq f_E \leq \chi_{[E]}$ (where $[E]$ is the clopen set in $X_{\mathcal{B}}$ corresponding to E), such that $\tilde{\mu}\{f_E < \lambda\} = \lambda\mu(E)$ for $0 \leq \lambda \leq 1$. Here $\tilde{\mu}$ corresponds to μ under the Stone correspondence.

(b) If \mathcal{B} is an F -algebra of subsets of a set X there is a \mathcal{B} -measurable function f_E on X with $0 \leq f_E \leq \chi_E$, for an $E \in \mathcal{B}$, such that $\mu\{f < \lambda\} = \lambda\mu(E)$ for $0 \leq \lambda \leq 1$.

PROOF. \mathcal{B} is an F -algebra iff $\{[E]: E \in \mathcal{B}\} = [\mathcal{B}]$ is an F -algebra of subsets of $X_{\mathcal{B}}$. The continuous functions on $X_{\mathcal{B}}$ as uniform limits of step functions based on clopen subsets of $X_{\mathcal{B}}$ are precisely the bounded $[\mathcal{B}]$ -measurable functions on $X_{\mathcal{B}}$. Thus, (a) follows from (b). To establish (b) one takes the family E_λ constructed in the proof of Theorem 2-2 and sets f_E equal to the uniform limit of $\{f_n: n \in N\}$ where f_n is the step function equal to $k \cdot 2^{-n}$ on $E_{(k+1)2^{-n}} \setminus E_{k2^{-n}}$ for $k < 2^n$.

REMARKS. (1) The proof of Theorem 2-2 is a simplification of Halmos' proof [10] of the convexity of the range of a countably-additive, finite-dimensional, nonatomic, vector-valued measure. Weiss [23] establishes Theorem 2-2 for the case $\mathcal{B} = 2^X$ for a set X . His proof closely follows Halmos' proof.

(2) Lemma 1-1 is a consequence of Theorem 2-2.

(3) Margolies [15] has shown that all left invariant means on infinite groups are nonatomic. Theorem 2-2 shows that the joint range, $(\mu_1(A), \dots, \mu_d(A))$ as A ranges over subsets of the group, of finitely many left invariant means (μ_1, \dots, μ_d) is convex. Results of Granirer [9] allow this result to be extended to left invariant means on right cancellative semigroups.

In the proof of Theorem 2-2 it is not necessary to use the full strength of the interpolation property defining F -algebras. It was noted by Margolies [15] that what is necessary is that if $\mu = (\mu_1, \dots, \mu_d)$, $\{A_n: n \in N\}$ and $\{B_n: n \in N\}$ are sequences in the algebra \mathcal{B} with $A_n \subset A_{n+1} \subset B_{n+1} \subset B_n$ for all n , and if j is such that $\mu_j^+(B_n \setminus A_n) \rightarrow 0$, or $\mu_j^-(B_n \setminus A_n) \rightarrow 0$, there exists a C in \mathcal{B} satisfying $A_n \subset C \subset B_n$ for all n . This property is called μ -completeness of \mathcal{B} . Then \mathcal{B} is μ -complete iff it is μ_j^+ and μ_j^- complete for all j .

The same definition may be used to define μ -complete subsets \mathcal{E} of \mathcal{B} . One may ask: "For which algebras \mathcal{B} does one have μ -completeness for all nonatomic μ ?"

If $A \subset B$ are in \mathcal{B} the restriction of a nonatomic μ to $\{C: C \in \mathcal{B}, C \subset B \setminus A\}$ is a nonatomic measure on this Boolean algebra. Hence $\{\mu(C): C \in \mathcal{B}, A \subset C \subset B\}$ is convex when the order interval $\langle A, B \rangle = \{C \in \mathcal{B}: A \subset C \subset B\}$ is μ -complete. If \mathcal{F} is a filter on \mathcal{B} and \mathcal{G} is an ideal then $\mathcal{F} \cap \mathcal{G}$ is an increasing union of order intervals $\langle A, B \rangle \subset \mathcal{F} \cap \mathcal{G}$. Thus we have the following corollary.

COROLLARY 2-2-2. *Let \mathcal{F} be a filter and \mathcal{G} be an ideal in the Boolean algebra \mathcal{B} . If $\mathcal{F} \cap \mathcal{G}$ is μ -complete for the nonatomic, finite-dimensional, finitely-additive measure μ then $\mu(\mathcal{F} \cap \mathcal{G}) = \{\mu(C): C \in \mathcal{F} \cap \mathcal{G}\}$ is convex.*

REMARK. Margolies has extended Theorem 2-2 to show that $\mu(\mathcal{E})$ is convex if \mathcal{E} is a μ -complete linear system in a Boolean algebra \mathcal{B} on which μ is nonatomic. A linear system is defined as a subset of \mathcal{B} so that the set of characteristic functions $\{\chi_{[A]}: A \in \mathcal{E}\}$ of the clopen sets $[A]$ in the Stone space of \mathcal{B} corresponding to elements A of \mathcal{E} are the $\{0, 1\}$ -valued elements in a linear variety of real functions on the Stone space. Linear systems \mathcal{E} are closed under complementation and disjoint union but not necessarily under intersection. Lack of closure under intersection causes our proof of Theorem 2-2 to break down. Given Margolies' result, an extension of Corollary 2-2-2 should be possible for affine systems \mathcal{E} where $\{\chi_{[A]}: A \in \mathcal{E}\}$ are the $\{0, 1\}$ -valued elements in an affine variety of functions on the Stone space of \mathcal{B} .

If \mathcal{B} is an infinite F -algebra and $\mu \in BA_1^+(\mathcal{B})$, the quotient \mathcal{B}_μ of \mathcal{B} modulo μ -negligible sets is an F -algebra which satisfies the countable chain condition. From this it may be shown that \mathcal{B}_μ is complete. This can be deduced from a result of Seever that the support of any measure on an F -space is Stonian [18, Theorem 2.2].

PROPOSITION 2-3. *If \mathcal{B} is an infinite F -algebra there exists a nonatomic $\mu \in BA_1^+(\mathcal{B})$. There are no infinite scattered F -spaces.*

PROOF. If not, the Stone space $X_{\mathcal{B}}$ is scattered and infinite, hence it contains a sequence $\{x_n: n \in N\}$ of distinct isolated points. Let Y be the closure of this sequence. Y is extremely disconnected and $\{x_n\}$ are isolated in Y . For any $A \subset N$, $Y_A = \overline{\{x_n: n \in A\}}$ is a clopen subset of Y . The mapping $A \rightarrow Y_A$ is an isomorphism of the clopen algebra of 2^N into the clopen algebra of Y . Since there is a nonatomic probability on 2^N there is one on the clopen algebra of Y . Hence Y has a nonatomic Radon probability, hence $X_{\mathcal{B}}$ has a nonatomic probability Radon measure, contradicting the scatteredness of $X_{\mathcal{B}}$. Thus, $X_{\mathcal{B}}$ is nonscattered. Thus, \mathcal{B} admits a nonatomic probability measure.

If X is an F -space which is infinite and scattered it is totally disconnected and is the Stone space of its F -algebra \mathcal{B} of clopen sets. Since \mathcal{B} admits a nonatomic probability, X could not have been scattered. \square

REMARK. This is established in [16, Corollary 1] for σ -complete algebras.

If \mathcal{B} is a σ -algebra and μ is countably additive then $\mathcal{R}(\mu)$ is compact. It is not known whether this is true if \mathcal{B} is an F -algebra. The following proposition states

that this is true iff all countably additive elements of $BA(\mathfrak{B})$ have Hahn decompositions.

PROPOSITION 2-4. *Let \mathfrak{B} be an F -algebra. Let F be a vector space of nonatomic measures on \mathfrak{B} . These two statements are equivalent.*

- (1) $\mathcal{R}(\mu)$ is compact if all components of μ are in F .
- (2) If $\mu \in F$ then μ has a Hahn decomposition.

PROOF. The implication (1) \rightarrow (2) is immediate. To see that (2) \rightarrow (1) note that we need only show that $\mathcal{R}(\mu)$ contains all extreme points of $\overline{\mathcal{R}(\mu)}$, for $\overline{\mathcal{R}(\mu)}$ is the convex hull of its extreme points and $\mathcal{R}(\mu)$ is convex. If x is an extreme point let l be a linear functional attaining its maximum on $\mathcal{R}(\mu)$ at x . The function $l \circ \mu: A \rightarrow l(\mu(A))$ is an element of F since all components of μ are in F . Since $l \circ \mu$ has a Hahn decomposition there is an $A \in \mathfrak{B}$ such that $l(\mu(A)) = l(x)$. We have $x \in \mathcal{R}(\mu)$. \square

If \mathfrak{B} is a Boolean algebra, and μ is a bounded, finitely-additive, finite-dimensional, nonatomic vector measure on \mathfrak{B} , let $\tilde{\mu}$ be the corresponding measure on $X_{\mathfrak{B}}$. If we set $\mathcal{R}'(\mu) = \{ \int f d\tilde{\mu} : f \in \mathcal{C}(X_{\mathfrak{B}}) : 0 \leq f \leq 1 \}$, then $\mathcal{R}'(\mu)$ is convex and $\mathcal{R}(\mu) \subset \mathcal{R}'(\mu) \subset \overline{\mathcal{R}(\mu)}$. These notations are understood in Propositions 2-5 and 2-6.

PROPOSITION 2-5. $\mathcal{R}'(\mu) = \text{conv}(\mathcal{R}(\mu))$.

PROOF. That $\text{conv}(\mathcal{R}(\mu)) \subset \mathcal{R}'(\mu)$ is immediate. If $x \in \mathcal{R}'(\mu) \setminus \text{conv}(\mathcal{R}(\mu))$ there is a linear functional l such that $l(x)$ is strictly larger than $l(y)$ if $y \in \text{conv}(\mathcal{R}(\mu))$. If $\mu_l = l \circ \mu$ then $\mu_l \in BA(\mathfrak{B})$ with $\mathcal{R}(\mu_l) = (-\|\mu_l^-\|, \|\mu_l^+\|)$ with $\|\mu_l^+\| = l(x)$. Since $x \in \mathcal{R}'(\mu)$ we have $l(x) \in \mathcal{R}'(\mu_l)$, so there is an $f \in \mathcal{C}(X_{\mathfrak{B}})$ with $\int f d\tilde{\mu}_l^+ = \|\mu_l^+\|$ and $0 \leq f \leq 1$. Since $\int f d\tilde{\mu}_l \leq \int f d\tilde{\mu}_l^+ \leq \|\mu_l^+\|$ we have $\int f d\tilde{\mu}_l^+ = \|\mu_l^+\|$ and $\int f d\tilde{\mu}_l^- = 0$. Thus, $\tilde{\mu}_l^+ \{f = 1\} = \|\mu_l^+\|$. Let $A \in \mathfrak{B}$ be such that $[A]$ is a clopen set containing $\{f = 1\}$ and disjoint from $\{f = 0\}$. We have $\mu_l^+(A) = \tilde{\mu}_l^+([A]) = \|\mu_l^+\|$ and $\mu_l^-(A) = \tilde{\mu}_l^-([A]) = 0$, so $\mu_l(A) = l(x)$, which is impossible. Hence, $\mathcal{R}'(\mu) \subset \text{conv}(\mathcal{R}(\mu))$, which establishes the proposition. \square

PROPOSITION 2-6 (LINDENSTRAUSS). *If $x \in \mathcal{R}'(\mu)$ is such that $\{f \in \mathcal{C}(X_{\mathfrak{B}}) : \int f d\tilde{\mu} = x\}$ has an extreme point f_0 , then $f_0 = \chi_{[A]}$ for some $A \in \mathfrak{B}$. Thus, $x \in \mathcal{R}(\mu)$.*

PROOF. Lindenstrauss [13]. \square

3. Ranges of two-dimensional, nonatomic, nonnegative measures with singular components. If $\mu = (\mu_1, \mu_2)$ is a nonatomic, nonnegative measure with singular components on an infinite F -algebra \mathfrak{B} , then $\mathcal{R}(\mu)$ is a convex subset of the closed rectangle $[0, \|\mu_1\|] \times [0, \|\mu_2\|]$ which contains the open rectangle $(0, \|\mu_1\|) \times (0, \|\mu_2\|)$ and the corners $(0, 0)$ and $(\|\mu_1\|, \|\mu_2\|)$. We will examine which convex sets in the closed rectangle are actually possible for such a μ . We will assume without loss of generality that $\|\mu_1\| = \|\mu_2\| = 1$ so we are dealing with convex subsets of the unit square $[0, 1] \times [0, 1]$. We remark that the involution $A \rightarrow A^c$ on \mathfrak{B} requires that $\mathcal{R}(\mu)$ be symmetric with respect to the center $(\frac{1}{2}, \frac{1}{2})$ of the square.

One possibility for $\mathfrak{R}(\mu)$ is that it be the entire closed square, which is true iff it contains either $(1, 0)$ or $(0, 1)$, hence the other. This is the case iff μ_1 and μ_2 assign full measure to disjoint elements of \mathfrak{B} . Such μ_1 and μ_2 exist on any infinite F -algebra by Proposition 2-3.

Is it possible for $\mathfrak{R}(\mu)$ to omit only $(1, 0)$ and $(0, 1)$? The answer is negative.

PROPOSITION 3-1. *If $\mu = (\mu_1, \mu_2)$ with μ_1 and μ_2 mutually singular nonatomic probabilities on the infinite F -algebra, it is impossible that $\mathfrak{R}(\mu)$ omits only $(0, 1)$ and $(1, 0)$ from the closed unit square.*

PROOF. Suppose that for every $\varepsilon > 0$ the points $(1 - \varepsilon, 0)$ and $(0, 1 - \varepsilon)$ belong to $\mathfrak{R}(\mu)$. We assert that there are disjoint $A(\varepsilon)$ and $B(\varepsilon)$ in \mathfrak{B} such that $\mu(A(\varepsilon)) = (1 - \varepsilon, 0)$ and $\mu(B(\varepsilon)) = (0, 1 - \varepsilon)$. Let $\mu(\tilde{A}(\varepsilon)) = (1 - \varepsilon, 0)$ and $\mu(\tilde{B}(\varepsilon)) = (0, 1 - \varepsilon)$. Set $A(\varepsilon) = \tilde{A}(\varepsilon) \setminus \tilde{B}(\varepsilon)$ and $B(\varepsilon) = \tilde{B}(\varepsilon) \setminus \tilde{A}(\varepsilon)$. Since $A(\varepsilon) \subset \tilde{A}(\varepsilon)$, $\mu_2(A(\varepsilon)) = 0$ and

$$\mu_1(A(\varepsilon)) = \mu_1(\tilde{A}(\varepsilon)) + \mu_1(\tilde{A}(\varepsilon) \cap \tilde{B}(\varepsilon)) = \mu_1(\tilde{A}(\varepsilon)) = 1 - \varepsilon.$$

Thus, $\mu(A(\varepsilon)) = (1 - \varepsilon, 0)$ and, similarly, $\mu(B(\varepsilon)) = (0, 1 - \varepsilon)$ with $A(\varepsilon)$ and $B(\varepsilon)$ disjoint. Suppose increasing sequences $\{A_k: k = 1, \dots, n\}$ and $\{B_k: k = 1, \dots, n\}$ have been constructed with $\mu(A_k) = (1 - \varepsilon^k, 0)$, $\mu(B_k) = (0, 1 - \varepsilon^k)$ and $A_k \cap B_k = \emptyset$ for all k , starting with $A_1 = A(\varepsilon)$, and $B_1 = B(\varepsilon)$. Let $A'_n = A_n \cup [A(\varepsilon^{n+1}) \setminus B_n]$ and $B'_n = B_n \cup [B(\varepsilon^{n+1}) \setminus A_n]$. We have $A'_n \cap B'_n = \emptyset$, $A_n \subset A'_n$, $B_n \subset B'_n$, $\mu_1(B'_n) = 0 = \mu_2(A'_n)$ and $\mu_1(A'_n) \geq 1 - \varepsilon^{n+1} \leq \mu_2(B'_n)$. There exist $A_n \subset A_{n+1} \subset A'_n$ and $B_n \subset B_{n+1} \subset B'_n$ with $\mu(A_{n+1}) = (1 - \varepsilon^{n+1}, 0)$ and $\mu(B_{n+1}) = (0, 1 - \varepsilon^{n+1})$. Thus, by induction, disjoint increasing sequences $\{A_n: n \in N\}$ and $\{B_n: n \in N\}$ are constructed with $\mu(A_n) = (1 - \varepsilon^n, 0)$ and $\mu(B_n) = (0, 1 - \varepsilon^n)$ for all n . Let $C_n = B_n^c$ so $\mu(C_n) = (1, \varepsilon^n)$ and $A_n \subset C_n$ for all n . Since \mathfrak{B} is an F -algebra there is a $D \in \mathfrak{B}$ with $A_n \subset D \subset C_n$ for all n . For this D we have $\mu(D) = (1, 0)$. Thus, if $\mathfrak{R}(\mu)$ contains $(1 - \varepsilon, 0)$ and $(0, 1 - \varepsilon)$ for all $\varepsilon > 0$ it contains $(1, 0)$ and $(0, 1)$. This completes the proof of this proposition. \square

The proof of Theorem 1-2 demonstrated that if μ_1 is a nonatomic probability measure on a Boolean algebra \mathfrak{B} with corresponding $\tilde{\mu}_1$ on the Stone space $X_{\mathfrak{B}}$, there is a nonatomic probability μ_2 on \mathfrak{B} with corresponding measure $\tilde{\mu}_2$ on $X_{\mathfrak{B}}$ such that $\text{supp}(\tilde{\mu}_2)$ is a nowhere dense subset of $\text{supp}(\tilde{\mu}_1)$ with $\tilde{\mu}_1(\text{supp}(\tilde{\mu}_2)) = 0$. Consequently, when $\mu_1(A) = 0$ then $\mu_2(A) = 0$ and there is an increasing sequence $\{A_n: n \in N\} \subset \mathfrak{B}$ such that $\mu_1(A) = 0$ converges to 1, yet $\mu_2(A_n) = 0$ for all n . If \mathfrak{B} is an F -algebra then $\mathfrak{R}(\mu)$ contains $(1 - \varepsilon, 0)$ for all $\varepsilon > 0$, yet contains no $(0, \lambda)$ with $\lambda > 0$. By averaging two such measures living on disjoint elements of \mathfrak{B} where one has had its coordinates reversed, one may obtain a μ such that $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, \lambda] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = \{0\} \times [0, 1 - \lambda]$ for $0 \leq \lambda \leq 1$. By averaging with yet another measure living on a disjoint element of \mathfrak{B} which has the closed unit square as its range, one may obtain a μ such that $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, \alpha] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = \{0\} \times (0, \beta)$ where $\{\alpha, \beta\} \subset [0, 1]$ and $\alpha + \beta \geq 1$. One may also find μ with $\mathfrak{R}(\mu) \cap ([0, 1] \times \{0\}) = [0, 1] \times \{0\}$ and $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = [0, \beta]$ or with $\mathfrak{R}(\mu) \cap (\{0\} \times [0, 1]) = [0, \beta)$ with $\beta < 1$.

Does there exist a μ on \mathfrak{B} so that $\mathcal{R}(\mu)$ is as small as possible so it consists of the open unit square $(0, 1) \times (0, 1)$ together with the corners $(0, 0)$ and $(1, 1)$? Equivalently, do there exist mutually singular nonatomic μ_1 and μ_2 such that if $A \in \mathfrak{B}$ then $\mu_1(A) > 0$ iff $\mu_2(A) > 0$? This latter question becomes, on the Stone space $X_{\mathfrak{B}}$, “Do there exist mutually singular nonatomic $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X_{\mathfrak{B}})$ with $\text{supp}(\tilde{\mu}_1) = \text{supp}(\tilde{\mu}_2)$?” We may replace $X_{\mathfrak{B}}$ by any compact Hausdorff space X and ask whether such $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X)$ exist. If $X = [0, 1]$ the existence of $\{\tilde{\mu}_1, \tilde{\mu}_2\}$ which are singular nonatomic and have $\text{supp}(\tilde{\mu}_1) = \text{supp}(\tilde{\mu}_2) = [0, 1]$ is well known.

PROPOSITION 3-2. *Let X be a nonscattered compact Hausdorff space. There exist $\{\tilde{\mu}_1, \tilde{\mu}_2\} \subset \mathfrak{M}_1^+(X)$ which are nonatomic, mutually singular and have the same support.*

PROOF. We may assume that X is perfect. Semadeni [19, 8.5.4 Theorem] shows that there is a continuous surjection $f: X \rightarrow [0, 1]$. Let Y be a minimal closed subset of X with $f(Y) = [0, 1]$. Thus, f is irreducible on Y . As a result, if $\tilde{\mu}$ is any measure on Y whose image, $\tilde{\mu} \circ f^{-1}$, under f has $[0, 1]$ as support then $\text{supp}(\tilde{\mu}) = Y$. If $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are measures on Y such that $\tilde{\mu}_1 \circ f^{-1} \perp \tilde{\mu}_2 \circ f^{-1}$ then $\tilde{\mu}_1 \perp \tilde{\mu}_2$. If $\tilde{\mu}$ is a measure on Y which has $\tilde{\mu} \circ f^{-1}$ nonatomic then $\tilde{\mu}$ is nonatomic. Since $[0, 1]$ has mutually singular nonatomic $\{v_1, v_2\} \subset \mathfrak{M}_1^+([0, 1])$ with $\text{supp}(v_i) = [0, 1]$ for $i = 1, 2$, we may find $\tilde{\mu}_i \in \mathfrak{M}_1^+(Y)$, via Hahn-Banach, with $\tilde{\mu}_i \circ f^{-1} = v_i$ for $i = 1, 2$. The measures $\{\tilde{\mu}_1, \tilde{\mu}_2\}$ are mutually singular nonatomic elements of $\mathfrak{M}_1^+(Y)$ with $\text{supp}(\tilde{\mu}_1) = Y = \text{supp}(\tilde{\mu}_2)$. This establishes the proposition. \square

COROLLARY 3-2-1. *There exists a family $\{\tilde{\mu}_t: t \in [0, 1]\} \subset \mathfrak{M}_1^+(X)$ such that $\tilde{\mu}_t \perp \tilde{\mu}_s$ if $t \neq s$, $\tilde{\mu}_t$ is nonatomic for all t and $\text{supp}(\tilde{\mu}_t) = \text{supp}(\tilde{\mu}_s)$ for all t, s .*

COROLLARY 3-2-2. *There exists on any Boolean algebra admitting a nontrivial nonatomic measure a family $\{\mu_t: t \in [0, 1]\} \subset BA_1^+(\mathfrak{B})$ such that $\mu_t \perp \mu_s$ if $t \neq s$, each μ_t is nonatomic and $\mu_t(A) = 0$ iff $\mu_s(A) = 0$ for all $A \in \mathfrak{B}$ and $\{t, s\} \subset [0, 1]$.*

For many algebras \mathfrak{B} much larger mutually singular families as in Corollary 3-2-3 are possible.

COROLLARY 3-2-3. *If \mathfrak{B} is an infinite F -algebra there exists a family $\{\mu_\alpha\} \subset BA_1^+(\mathfrak{B})$ of mutually singular nonatomic measures of cardinality 2^c with the property that if $A \in \mathfrak{B}$ then $\mu_\alpha(A) = 0$ iff $\mu_\beta(A) = 0$ for all α and β .*

PROOF. There exists a family $\{\nu_\alpha\}$ of mutually nonatomic Radon probability measures on $\{0, 1\}^c$ of cardinality 2^c each with support equal to $\{0, 1\}^c$. This may be constructed by using, for $\{\nu_\alpha\}$, coin flip measures with c flips and with varying probabilities of heads. All that is needed to establish the corollary is to find a closed subset of $X_{\mathfrak{B}}$ admitting $\{0, 1\}^c$ as a continuous image. To do this pick a nonatomic element of $\mathfrak{M}^+(X_{\mathfrak{B}})$ and let Z be its support. Seever [18] shows that Z is Stonian, hence has a complete (infinite) clopen algebra \mathfrak{B}_Z . One may find a disjoint sequence in $\mathfrak{B}_Z \setminus \{\emptyset\}$. The σ -complete algebra \mathfrak{B}_0 it generates is isomorphic to 2^N . There is, as a result, a continuous map from Z onto βN . Since $\{0, 1\}^c$ is

separable (Comfort and Negrepontis [6, Theorem 3.20]) there is a continuous map of βN onto $\{0, 1\}^c$. Thus, Z has $\{0, 1\}^c$ as a continuous image. \square

REMARK. It is possible to choose the measures in Corollary 3-2-4 so that if $\{\mu_{n_1}: n \in N\}$ and $\{\mu_{n_2}: n \in N\}$ are disjoint subsets there is an $A \subset N$ such that $\mu_{n_1}(A) \geq 1 - \varepsilon$ and $\mu_{n_2}(A) \leq \varepsilon$ for all n for a given $\varepsilon > 0$. This requires the use of some results from the theory of independent sets.

One example of an F -algebra is gotten by taking the power set 2^κ of a cardinal κ and the ideal $[\kappa]^{<\kappa}$ of subsets of cardinality less than κ . The quotient algebra $\mathfrak{B} = 2^\kappa / [\kappa]^{<\kappa}$ is an F -algebra which is not σ -complete when κ is of countable cofinality. The measures on \mathfrak{B} are precisely those arising from uniform measures on κ (i.e., measures which annihilate all subsets of κ of smaller cardinal).

COROLLARY 3-2-4. *There is, for any infinite cardinal, κ , a collection $\{\mu_t\}$ of cardinality 2^c of uniform measures on κ which are nonatomic, have $\mu_t(\kappa) = 1$ for all t , are mutually singular and have the property that if $A \subset \kappa$, then $\mu_t(A) = 0$ iff $\mu_s(A) = 0$ for all s and t .*

PROPOSITION 3-3. *Let \mathfrak{B} be an infinite F -algebra. The only convex subset A of $[0, 1] \times [0, 1]$ symmetric about $(\frac{1}{2}, -\frac{1}{2})$ which contains $((0, 1, 1) \times (0, 1)) \cup \{(0, 0), (1, 1)\}$ which is not $\mathcal{R}(\mu)$ for some $\mu = (\mu_1, \mu_2)$ with μ_1 and μ_2 mutually singular nonatomic elements of $BA^+(\mathfrak{B})$ is $([0, 1] \times [0, 1]) \setminus \{(0, 1), (1, 0)\}$.*

PROOF. This is easily verified based on Proposition 3-1, Corollary 3-2-2 and remarks in this section. We only remark that to obtain $((0, 1) \times (0, 1)) \cup ([0, \lambda] \times \{0\}) \cup ([1 - \lambda, 1] \times \{1\})$ one takes a pair $(\mu_1, \mu_2) = \mu$ with

$$\mathcal{R}(\mu) = ((0, 1) \times (0, 1)) \cup \{(0, 0), (1, 1)\}$$

by Corollary 3-2-2 then one picks an $A_0 \in \mathfrak{B}$ with $\mu_1(A_0) = \lambda = \mu_2(A_0)$. Finally set

$$\nu(A) = (\nu_1(A_1), \nu_2(A)) = (\mu_1(A), \mu_2(A \setminus A_0)(1 - \lambda)^{-1})$$

for $A \in \mathfrak{B}$. \square

Consider the case of $\mu = (\mu_1, \dots, \mu_n)$ with $\{\mu_1, \dots, \mu_n\}$ mutually singular nonatomic probability measures. $\mathcal{R}(\mu)$ is dense in $[0, 1]^n$ and is convex if μ is defined on an F -algebra, hence contains $(0, 1)^n$, $(0, \dots, 0)$, and $(1, \dots, 1)$. Use of Corollary 3-2-3 shows that it is possible that $\mathcal{R}(\mu)$ contains only the points $(0, \dots, 0)$ and $(1, \dots, 1)$ on the boundary. It is also possible that $\mathcal{R}(\mu)$ be all of $[0, 1]^n$. In general, to examine $\mathcal{R}(\mu)$ requires the examination of $\mathcal{R}(\mu) \cap F$ for faces F of the cube $[0, 1]^n$. From the results of [2, Lemma 3] a face F of $[0, 1]^n$ is of the form $\square_{A,B}^+ = \{(x_1, \dots, x_n): x_i = 0 \text{ if } i \notin B \text{ and } x_i = 1 \text{ if } i \in A\}$. $E \in \mathfrak{B}$ has $\mu(E) \in F$ iff $\mu_i(E) = 0$ if $i \notin B$ and $\mu_i(E) = 1$ if $i \in A$. Thus, if \mathcal{G}_B is the ideal $\{E \in \mathfrak{B}: \mu_i(E) = 0 \text{ if } i \notin B\}$ and \mathcal{F}_A is the filter $\{E \in \mathfrak{B}: \mu_i(E) = 1 \text{ if } i \in A\}$, then $\mathcal{R}(\mu) \cap F = \mu(\mathcal{G}_B \cap \mathcal{F}_A)$.

PROPOSITION 3-4. *Let $\mu = (\mu_1, \dots, \mu_n)$ with μ_1, \dots, μ_n mutually singular probability measures on the F -algebra \mathfrak{B} . For any face F of $[0, 1]^n$ there is a closed order interval (for the usual order on R^n) $\langle y_F, x_F \rangle \subset F$ so that $\mathcal{R}(\mu) \cap F$ is a dense (for the relative topology on F) subset of $\langle y_F, x_F \rangle$. Furthermore if $F \subset G$ are faces of $[0, 1]^n$ then $\langle y_F, x_F \rangle \subset \langle y_G, x_G \rangle$.*

PROOF. If $F \cap \mathcal{R}(\mu) = \mu(\mathcal{F} \cap \mathcal{G})$ for a filter \mathcal{F} and an ideal \mathcal{G} , let $x_F = \sup\{\mu(A): A \in \mathcal{F} \cap \mathcal{G}\}$ and $y_F = \inf\{\mu(A): A \in \mathcal{F} \cap \mathcal{G}\}$ (for the order of R^n). Note that if $A_1 \subset A_2$ are in $\mathcal{F} \cap \mathcal{G}$ then $\mathcal{R}(\mu) \cap F$ contains the relative interior of the order interval $\langle \mu(A_1), \mu(A_2) \rangle$ since μ has singular components on $A_2 \setminus A_1$. As A_1 decreases and A_2 increases in $\mathcal{F} \cap \mathcal{G}$ these relative interiors increase to cover the relative interior of $\langle y_F, x_F \rangle$. This suffices to show that $\mathcal{R}(\mu) \cap F$ is a dense subset of $\langle y_F, x_F \rangle$. That, when $F \subset G$, $\langle y_F, x_F \rangle \subset \langle y_G, x_G \rangle$ is an immediate consequence of the convexity of $\mathcal{R}(\mu)$. \square

REMARKS. In essence Proposition 3-4 is an analysis of the facial structure of $\mathcal{R}(\mu)$. If one examines relative interiors of faces $\mathcal{R}(\mu)$ one obtains the Gleason parts of $\mathcal{R}(\mu)$. Corollary to Proposition 3-4 we may say that the Gleason parts of $\mathcal{R}(\mu)$ are all "open rectangles". Thus, in the n -dimensional case $\mathcal{R}(\mu)$ is far from being an arbitrary convex subset of $[0, 1]^n$ symmetric about $(\frac{1}{2}, \dots, \frac{1}{2})$ and containing $(0, \dots, 0)$ and $(1, \dots, 1)$.

4. Zonoids. In §2 we exhaustively considered the possible ranges of a two-dimensional, finitely-additive vector measure on an F -algebra with mutually singular probabilities as components. The conclusion was that "nearly all" dense convex subsets of the unit square containing $(0, 0)$ and $(1, 1)$ are possible. We conjecture that the same sort of results are true in general for nonatomic vector valued measures with nonsingular components.

We recall from [4] that a *zonoid* is the range of a finite-dimensional, nonatomic, countably-additive, vector-valued measure on a measurable space (X, Σ) . Lemma 1-1 assures us that the closure of the range of a finite-dimensional, finitely-additive, nonatomic measure on a Boolean algebra is always a zonoid. A converse also is true as we shall see in Proposition 4-2. We shall assume that all zonoids are in \mathbf{R}^d for some d . When a zonoid lies in the nonnegative orthant of \mathbf{R}^d we shall call it a positive zonoid. The countably-additive, vector-valued, nonatomic measures giving rise to positive zonoids are the nonnegative ones.

In [4] it is shown that zonoids in \mathbf{R}^d are precisely the class of weak* $(\sigma(l^\infty, l^1))$ continuous linear images in \mathbf{R}^d of the unit ball, \square_∞ , of l^∞ . Since \square_∞ is a $\sigma(l^\infty, l^1)$ continuous linear image of the positive unit ball, $\square_\infty^+ = [0, 1]^{\aleph_0}$, all zonoids are $\sigma(l^\infty, l^1)$ continuous linear images of \square_∞^+ . This is reasonable for \square_∞^+ is an "infinite-dimensional zonoid". If $\mu = \{\mu_n: n \in N\}$ is a sequence of mutually singular, countably-additive, nonatomic probability measures on a measurable space (X, Σ) , then μ may be considered a nonnegative, l^∞ -valued, nonatomic vector measure by setting $\mu(A) = (\mu_n(A))$. Then $\mathcal{R}(\mu)$ is easily verified to be \square_∞^+ . If $\tau: l^\infty \rightarrow \mathbf{R}^d$ is $\sigma(l^\infty, l^1)$ continuous and linear then $\tau \circ \mu$ is a finite-dimensional, nonatomic, countably-additive measure on (X, Σ) with range $\tau(\square_\infty^+)$. Note that τ may be described by a matrix $\{t_{ij}: 1 \leq i \leq d, 1 \leq j < \infty\}$ so that $\tau((x_n))$ has its i th coordinate given by $\sum_{j=1}^\infty t_{ij}x_j$. For each $1 \leq i \leq d$, $\{t_{ij}: 1 \leq j < \infty\} \in l^1$. It may be assumed that for each j in N there is an i with $t_{ij} \neq 0$. The zonoid $\tau(\square_\infty^+)$ is a positive zonoid iff $\{t_{ij}: 1 \leq j < \infty\} \in l^1_+$ for each $1 \leq i \leq d$. The fact that $\tau \circ \mu$ is nonatomic is a consequence of the fact that countable convex combinations of nonatomic measures (countably additive or finitely additive) are nonatomic. The countable additivity of $\tau \circ \mu$ is established similarly.

PROPOSITION 4-1. *Let μ be a finitely-additive map from the Boolean algebra \mathfrak{B} into the locally convex space E with dual E' . Let $l \circ \mu$ be a nonatomic element of $BA(\mathfrak{B})$ for all $l \in E'$. The $\sigma(E, E')$ closure of $\mathfrak{R}(\mu)$ is convex and $\sigma(E, E')$ bounded.*

PROOF. Boundedness of $\mathfrak{R}(\mu)$ is immediate. Let C be the closed convex hull of $\mathfrak{R}(\mu)$ for $\sigma(E, E')$. For any $\{l_1, \dots, l_n\} \subset E'$ consider $(l_1, \dots, l_n) \circ \mu: \mathfrak{B} \rightarrow R^n$. Let $\mathfrak{R}(l_1, \dots, l_n)$ be the set of $e \in E$ such that $(l_1(e), \dots, l_n(e))$ is in the closure of the range of $(l_1, \dots, l_n) \circ \mu$. $\mathfrak{R}(l_1, \dots, l_n)$ is $\sigma(E, E')$ closed and convex by Lemma 1-1, hence contains C . The intersection D , of all $\mathfrak{R}(l_1, \dots, l_n)$ as $\{l_1, \dots, l_n\}$ ranges over finite subsets of E' , contains C . If $e \in D$ there is, for any $\{l_1, \dots, l_n\} \subset E'$ and $\varepsilon > 0$, an $A \in \mathfrak{B}$ with $|l_i(\mu(A)) - l_i(e)| < \varepsilon$ for $i = 1, \dots, n$. Thus, D is in the closure of $\mathfrak{R}(\mu)$ for $\sigma(E, E')$. Hence $D = C$, which establishes the proposition. \square

PROPOSITION 4-2. *Let Z be a zonoid and \mathfrak{B} a Boolean algebra with a nontrivial nonatomic measure. There is a finitely-additive, nonatomic measure ν on \mathfrak{B} such that $\overline{\mathfrak{R}(\nu)} = Z$.*

PROOF. Let τ be a $\sigma(l^\infty, l^1)$ continuous function so that $\tau(\square Z_\infty^+) = Z$. Let $\{\mu_n: n \in N\}$ be a sequence of mutually singular, nonatomic, finitely-additive probability measures on \mathfrak{B} given by Corollary 1-2-3. Let μ be the l^∞ -valued $\sigma(l^\infty, l^1)$ nonatomic measure on \mathfrak{B} given by $\mu(A) = (\mu_n(A))$. By Lemma 4-1 $\mathfrak{R}(\mu)$ is $\sigma(l^\infty, l^1)$ dense in \square_∞^+ . Let $\nu = \tau \circ \mu$. It is immediate that $\mathfrak{R}(\nu)$ is dense in Z (the topology on Z being that induced by its linear span so $Z^0 \neq \emptyset$). This establishes the proposition. \square

PROPOSITION 4-3. *Let Z be a zonoid and \mathfrak{B} an F -algebra. There is a nonatomic ν on \mathfrak{B} such that $\mathfrak{R}(\nu)$ is $Z^0 \cup \{\nu(\emptyset), \nu(X)\}$.*

PROOF. Use Corollary 3-2-2 to construct a sequence $\{\mu_n: n \in N\}$ of mutually singular, nonatomic, finitely-additive probabilities on \mathfrak{B} such that these all have identical null sets. Construct the nonatomic l^∞ -valued measure μ as in the proof of Proposition 4-2. Then, $\mathfrak{R}(\mu)$ is $\sigma(l^\infty, l^1)$ dense in \square_∞^+ . The only sequence $\{x_n: n \in N\}$ in $\mathfrak{R}(\mu)$ with $x_n = 0$ for some n is 0, and the only sequence in $\mathfrak{R}(\mu)$ with $x_n = 1$ for some n is 1. Let τ be a $\sigma(l^\infty, l^1)$ continuous linear function with $\tau(\square_\infty^+) = Z$. Let $\nu = \tau \circ \mu$. Since $\mathfrak{R}(\nu)$ is dense in Z by Proposition 4-2 and is convex by Proposition 2-4, it includes $Z^0 \cup \{\nu(\emptyset), \nu(X)\}$ (which is convex). If $0 \in \partial Z$ let F be the minimal face of Z containing 0. $\tau^{-1}(F)$ is a $\sigma(l^\infty, l^1)$ closed face of \square_∞^+ which must be proper since $\tau(1) = \nu(X)$, and since the face of Z determined by 0 and $\nu(X)$ is Z and not $F \subset \partial Z$. There are no elements of $\mathfrak{R}(\nu)$ in $F \setminus \{0\}$, for if there were there would be a sequence $(\mu_n(A)) \in \tau^{-1}(F)$. By [2, Lemma 3 (6)], $\tau^{-1}(F)$ is of the form $\{(x_n): x_n = 0 \text{ for all } n \in J\}$ for some nonempty subset J of N . If $\mu(A) \in \tau^{-1}(F)$ then $\mu_n(A) = 0$ for some $n \in N$, hence $\mu_n(A) = 0$ for all $n \in N$, hence $\mu(A) = 0$, hence $\nu(A) = 0$. Thus, $\{0\} = F \cap \mathfrak{R}(\nu)$. Similarly, if F' is the smallest face of Z containing $\nu(N)$ then $\mathfrak{R}(\nu) \cap F' = \{\nu(N)\}$. Now consider $x \in \partial Z$ such that the smallest face F of Z containing x does not contain either 0 or $\nu(X)$ (which is the case if $\{0, \nu(X)\} \subset Z^0$). $\tau^{-1}(F)$ is a

$\sigma(l^\infty, l^1)$ closed face of \square_∞^+ . Hence there are, by [2, Lemma 3 (6)], subsets $J_1 \subset J_2$ of N so that (x_n) in \square_∞^+ is in $\pi^{-1}(F)$, $x_n = 0$ if $n \notin J_2$, and $x_n = 1$ if $n \in J_1$. It must be the case that $J_2 \neq N$ and $J_1 \neq \emptyset$. It is impossible to have an $A \in \mathfrak{B}$ so that $\mu_n(A) = 0$ for $n \notin J_2$ and $\mu_n(A) = 1$ if $n \in J_1$. Thus, $\mathfrak{R}(\mu) \cap \tau^{-1}(F) = \emptyset$. Thus, $\mathfrak{R}(\nu) \cap F = \emptyset$. This suffices to establish the proposition. \square

REMARK. It immediately follows that if \mathfrak{B} is any algebra supporting a nonatomic measure, there is a nonatomic vector-valued measure ν on \mathfrak{B} whose range is dense in Z yet contains no boundary points of Z other than 0 and $\nu(X)$.

In the proof of Proposition 4-3 it was observed that if F is a face of the zonoid Z then $\tau^{-1}(F)$ is a $\sigma(l^\infty, l^1)$ closed face \square_∞^+ , hence of the form

$$\square_{A(F), B(F)}^+ = \{(x_n): \chi_{A(F)} \leq (x_n) \leq \chi_{B(F)}\}$$

by [2, Lemma 3 (6)], where $A(F) \subset B(F)$ are subsets of N . If F_1 and F_2 are faces of Z then

$$A(F_1 \cap F_2) = A(F_1) \cup A(F_2) \quad \text{and} \quad B(F_1 \cap F_2) = B(F_1) \cap B(F_2).$$

F_1 and F_2 are disjoint iff $A(F_1) \setminus B(F_2) \neq \emptyset$ or $A(F_2) \setminus B(F_1) \neq \emptyset$ so $A(F_1 \cap F_2) \subsetneq B(F_1 \cap F_2)$. When $F_1 \neq F_2$ are zero dimensional (i.e. extreme points) then F_1 and F_2 are disjoint.

If one wishes to find a nonatomic ν so that $\mathfrak{R}(\nu) \cap F \neq \emptyset$ for a given face F of Z with $\nu = \tau \circ \mu$ as in the proof of Proposition 4-3, it is necessary to guarantee that μ_n for $n \in A(F)$ be simultaneously disjoint with μ_n for $n \notin B(F)$ so that there is an $A \in \mathfrak{B}$ so that $\mu_n(A) = 1$ for $n \in A(F)$ and $\mu_n(A) = 0$ if $n \notin B(F)$.

PROPOSITION 4-4. *Let \mathfrak{B} be an algebra with a nonzero nonatomic measure. Let Z be a zonoid. Let $\{F_1, \dots, F_n\}$ be faces of Z . There is a nonatomic vector-valued measure ν with $\mathfrak{R}(\nu)$ dense in Z with $\mathfrak{R}(\nu) \cap F_j \neq \emptyset$ for $j = 1, \dots, n$ and so that there are at most finitely many faces F of Z with $\mathfrak{R}(\nu) \cap F \neq \emptyset$.*

PROOF. Let $\{C_1, \dots, C_m\}$ be the atoms of the finite algebra generated by $\{A(F_j): 1 \leq j \leq n\}$ in 2^N . Let $\{A_1, \dots, A_m\}$ be the atoms of a finite subalgebra \mathcal{Q} of \mathfrak{B} so that each A_i receives strictly positive measure under some nonatomic measure on \mathfrak{B} . Select for each $n \in C_j$ a nonatomic probability μ_n on \mathfrak{B} such that $\mu_n(A_j) = 1$ so that $\{\mu_n: n \in C_j\}$ are mutually singular and have identical null sets. Let $\mu = (\mu_n)$ and let $\nu = \tau \circ \mu$. Since μ has mutually singular probabilities as components, $\mathfrak{R}(\nu)$ is dense in Z . For any j , $A(F_j)$ is a union of a subset of $\{C_1, \dots, C_m\}$, say $\{C_{j_1}, \dots, C_{j_l}\}$. If $n \in A(F_j)$ then $\mu_n(A_{j_1} \cup \dots \cup A_{j_l}) = 1$. If $n \notin B(F_j)$ then $n \notin A(F_j)$ so $\mu_n(A_{j_1} \cup \dots \cup A_{j_l}) = 0$. Thus, $\{\mu_n: n \in A(F_j)\}$ are mutually disjoint with $\{\mu_n: n \notin B(F_j)\}$. Thus, $\mathfrak{R}(\nu) \cap F_j \neq \emptyset$. Now suppose that F is a face of Z with $\mathfrak{R}(\nu) \cap F \neq \emptyset$. Let $A \in \mathfrak{B}$ with $\nu(A) \in F$ so that $(\mu_n(A)) \in \tau^{-1}(F)$. If $n \in C_j$ and $\mu_n(A) = 1$ then $\mu_n(A \cap A_j) = 1$ so $\mu_m(A \cap A_j) = 1$ for all $m \in C_j$, hence $\mu_m(A) = 1$ for all $m \in C_j$. Thus, $C_j \subset A(F)$ if $C_j \cap A(F) \neq \emptyset$. That is, $A(F)$ is a union of the C_j 's, hence is in \mathcal{Q} . Similarly the complement of $B(F)$ is in \mathcal{Q} . Since F is determined by $A(F)$ and $B(F)$ and since there are only finitely many $A(F) \subset B(F)$ in \mathcal{Q} , only finitely many faces F of Z have $F \cap \mathfrak{R}(\nu) \neq \emptyset$. \square

REMARK. If $r = 2^n$ an upper bound for the number of faces F of Z meeting $\mathcal{R}(\nu)$ is

$$\sum_{j=0}^r \left[C(r, j) \sum_{k=0}^j C(j, k) \right] \leq (2^r)^2.$$

PROPOSITION 4-5. Let \mathcal{B} be an infinite F -algebra. Let Z be a zonoid. There is a nonatomic ν on \mathcal{B} with $\mathcal{R}(\nu) = Z$.

PROOF. Let μ_0 be a nonatomic probability \mathcal{B} . The measure algebra \mathcal{B}_{μ_0} is complete. Let $\{A_n\}$ be a partition of \mathcal{B}_{μ_0} with nonempty elements. Let μ_n be the probability measure on \mathcal{B}_{μ_0} obtained by restricting μ_0 (considered as a probability on \mathcal{B}_{μ_0}) to A_n and normalizing. The sequence $\{\mu_n: n \in N\}$ consists of mutually disjoint probabilities. If $J \subset N$ then $\mu_n(\cup \{A_m: m \in A(F)\}) = 1$ and $\mu_n(\cup \{A_m: m \notin J\}) = 0$ if $n \in J$. Let $\nu = \tau \circ \mu$ be defined on \mathcal{B}_{μ_0} , hence on \mathcal{B} in the usual fashion. Let F be a face of Z . We have $\mu(\cup \{A_m: m \in A(F)\}) = \chi_{A(F)} \in \tau^{-1}(F)$ so $\nu(\cup \{A_m: m \in A(F)\}) \in F$. Thus, $\mathcal{R}(\nu) \cap F \neq \emptyset$ for all faces F of Z . In particular, $\mathcal{R}(\nu)$ contains all extreme points of Z . Since $\mathcal{R}(\nu)$ is convex, $Z = \mathcal{R}(\nu)$.

□

If μ is a finitely-additive nonatomic measure on a Boolean algebra then $\overline{\mathcal{R}(\mu)}$ is a zonoid Z . If F is a face of Z then F is the intersection of a supporting hyperplane H with Z and H may be considered as a supporting hyperplane to $\mathcal{R}(\mu)$. Conversely if H is a supporting hyperplane to $\mathcal{R}(\mu)$ it is a supporting hyperplane to Z . If $\mathcal{R}(\mu)$ is convex its faces are the traces on $\mathcal{R}(\mu)$ of the faces of Z .

PROPOSITION 4-6. If μ is a nonatomic measure on a Boolean algebra \mathcal{B} and H is a supporting hyperplane to $\mathcal{R}(\mu)$ then $H \cap \mathcal{R}(\mu)$ is $\mu(\mathcal{F}_H \cap \mathcal{G}_H)$ where \mathcal{F}_H is a filter and \mathcal{G}_H is an ideal on \mathcal{B} . Furthermore, $H \cap \mathcal{R}(\mu)$ is a zonoid or a translate of a zonoid.

PROOF. Let l be a linear functional so that H is the affine variety $\{l = c\}$ so that $\mathcal{R}(\mu) \subset \{l \leq c\}$. Then $c = \|l \circ \mu\|$ and $l(\mu(A)) \leq c$ for $A \in \mathcal{B}$. If $\mu(A) \in H$ then $\mu(A) = c$. Let $\mathcal{C} = \{A: l(\mu(A)) = c\}$. If $\{A_1, A_2\} \subset \mathcal{C}$ then

$$2c = l(\mu(A_1)) + l(\mu(A_2)) = l(\mu(A_1 \cup A_2)) + l(\mu(A_1 \cap A_2)) \leq 2c.$$

Consequently, $\{A_1 \cup A_2, A_1 \cap A_2\} \subset \mathcal{C}$. Thus, \mathcal{C} must be of the form $\mathcal{F}_H \cap \mathcal{G}_H$ for some filter \mathcal{F}_H and ideal \mathcal{G}_H in \mathcal{B} .

Choose $\{A_n\}$, a decreasing sequence in \mathcal{C} , and $\{B_n\}$, an increasing sequence in \mathcal{C} , with $A_n \subset B_n$ for all n and with $\overline{H \cap \mathcal{R}(\mu)}$ equal to the closure of $\bigcup_{n=1}^{\infty} \{\mu(C): A_n \subset C \subset B_n\}$. By passage to a subsequence assume that $\mu(A_n)$ converges to x_0 . Then $\{\mu(C): A_n \subset C \subset B_n\} - \mu(A_n)$ are zonoids which converge to $\overline{H \cap \mathcal{R}(\mu)} - x_0$ in the Hausdorff metric. Results of Bolker [4] show that $\overline{H \cap \mathcal{R}(\mu)} - x_0$ is a zonoid. This establishes the proposition. □

REMARKS ADDED IN PROOF. (1) Peter Loeb in *A combinatorial analog of Lyapounov's theorem for infinitesimally generated atomic vector measures*, Proc. Amer. Math. Soc. 39 (1973), 585–586, establishes a result which has been used by Donald Brown to establish Lemma 1-1.

(2) Domenico Candeloro and Anna Martellotti Sacchetti in *Sul rango di una massa vettoriale*, Atti. Sem. Mat. Fis. Univ. Modena **28** (1979), 102–111, establish Theorem 2-2 for $\mathfrak{B} = 2^X$ using Halmos' proof.

(3) F. Dashiell, A. Hager and M. Henriksen, in *Order-Cauchy completions of rings and vector lattices of continuous functions*, Canad. J. Math. **32** (1980), 657–685, have introduced the notion of a quasi- F -space X which, for compact X , is one for which $C(X)$ is sequentially order-Cauchy complete. If X is the Stone space of a Boolean algebra \mathfrak{B} then \mathfrak{B} should be called a quasi- F -algebra iff X is a quasi- F -space. This is the case iff whenever $\{A_n\}$ is an increasing sequence and $\{B_n\}$ is a decreasing sequence in \mathfrak{B} with $A_n \subset B_n$ for all n and for which $\{B_n \setminus A_n\}$ decreases to \emptyset there is a $C \in \mathfrak{B}$ with $A_n \subset C \subset B_n$ for all n . In the results of this article in which F -algebras appear one may use quasi- F -algebras instead. This is clear from remarks after Corollary 2-2-1. It is useful to note that it follows from Corollary 4.7 of Dashiell, Hager and Henriksen that, just as for F -spaces, the support of any Radon measure or a compact quasi- F -space is Stonian.

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